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LETTER TO THE EDITOR

On the Painlevé property of the SO(2, 1) invariant non-linear σ model

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Abstract. It is shown that the (1+1)-dimensional SO(2, 1) invariant non-linear σ model possesses the Painlevé (P) property. Consequently the nature of the Bäcklund transformation is discussed.

In recent years considerable interest has been focused on the investigation of completely integrable non-linear dynamical systems (Ablowitz and Segur 1981). Ablowitz *et al* (1980), presented a possible method in which they pointed out the connection between the non-linear partial differential equations (PDEs) solvable by the inverse scattering transform (IST) method and non-linear ordinary differential equations (ODEs) without moveable critical points. It is well known that an ODE is said to have Painlevé (P) property (or type) if the movable singularities exhibited by the general solution are only poles in the complex plane. They have also conjectured that every non-linear ODE obtained by an exact reduction of a completely integrable non-linear PDE is of P type. The P analysis has been successfully carried out for a large class of non-linear dynamical systems (Ablowitz *et al* 1980, Lakshmanan and Kaliappan 1983, Lakshmanan and Sahadevan 1985) to identify their integrability.

More recently, Weiss *et al* (1983) proposed a direct method without recourse to the reduction to ODEs, generalising the corresponding P property for the PDEs. A modified definition is as follows. A PDE is said to be of P type if its solution can be expressed as a 'single-valued' expansion about its non-characteristic movable singularity manifold. More precisely, if the singularity manifold is determined by

$$\phi(x, t)=0, \qquad \phi_x(x, t)\neq 0, \tag{1}$$

and if $u=u(x, t)$ is a solution of the PDE, then we assume that

$$u=\phi^{\alpha}\sum_{j=0}^{\infty} u_j\phi^j, \tag{2}$$

where

$$\phi=\phi(x, t), \qquad u_j=u_j(x, t), \qquad u_0\neq 0$$

are analytic functions of (x, t) in a neighbourhood of the manifold (1) and α is an integer. It is demonstrated that this method is a useful tool to test the integrability of the PDEs by various authors (Weiss *et al* 1983, Chudnovsky *et al* 1983, Steeb *et al* 1984, Ward 1984, Weiss 1984). In this letter we apply this method to the SO(2, 1) invariant non-linear σ model in $(1+1)$-spacetime dimension and show that it possesses the P property.

By a suitable parametrisation, the equation of motion of the above model takes the form (Chinea 1981)

$$Z_{xt}-2Z_xZ_t/(Z+\bar{Z})=0, \tag{3}$$

where $\bar{Z}$ denotes the complex conjugate of Z and subscripts denote the partial differentiations. It is well known that the above type of model plays an important role in elementary particle physics (Pohlmeyer 1976, Lund 1976, Lakshmanan and Tamizhmani 1981). Now, substituting $Z=u+\mathrm{i}v$, u and v are real, in (3), we obtain

$$uu_{xt}-u_xu_t+v_xv_t=0, \qquad uv_{xt}-u_tv_x-u_xv_t=0. \tag{4a, b}$$

The Painlevé test for PDE also proceeds along three main stages in the same way as in the case of ODEs: (i) determination of leading-order behaviours, (ii) identifying the powers at which the arbitrary functions can enter, called resonances, and (iii) verifying that a sufficient number of arbitrary functions exist without the introduction of algebraic and logarithmic singularities.

We assume that the leading orders of the solutions of (4) behave as

$$u\approx u_0\phi^{\alpha}, \qquad v\approx v_0\phi^{\beta}, \tag{5}$$

where u_0, v_0 are analytic functions of (x, t), α and β are integers to be determined. Making use of (5) in (4), by equating the most singular terms, we obtain

$$\alpha=\beta=-1 \qquad \text{(say)}, \tag{6a}$$

and hence

$$u_0^2+v_0^2=0, \qquad 0\cdot u_0v_0\phi_x\phi_t=0. \tag{6b}$$

From $(6b)$, we infer that the coefficient v_0 (say) is arbitrary and so,

$$u_0=\mathrm{i}\varepsilon v_0, \qquad \varepsilon=\pm 1. \tag{7}$$

Having obtained this leading-order behaviour, we look at the powers (resonances) at which the arbitrary functions enter. For this purpose, we substitute the following series representation

$$u\approx u_0\phi^{-1}+\sum_{j=1}^{\infty}u_j\phi^{j-1}, \qquad v\approx v_0\phi^{-1}+\sum_{j=1}^{\infty}v_j\phi^{j-1} \tag{8a, b}$$

into (4), retaining the leading-order terms. Equating the coefficients of ϕ^{j-4} $(j>0)$, we obtain a pair of equations in (u_j, v_j) which may be conveniently written in the form

$$\begin{bmatrix} 2j & \mathrm{i}(j^2-j) \\ \mathrm{i}(j^2-j+2) & -2(j-1) \end{bmatrix}\begin{bmatrix} u_j \\ v_j \end{bmatrix}=0. \tag{9}$$

From (9), we obtain the resonances

$$j=-1, 0, 1, 2. \tag{10}$$

Obviously, the resonance value at $j=-1$ represents the arbitrariness of the singularity manifold $\phi(x, t)=0$, while the resonance $j=0$ is associated with the arbitrariness of v_0, as seen in $(6b)$.

Making use of the series representation (8) in (4) and comparing the coefficients of (ϕ^{-3}, ϕ^{-3}) we obtain an identical equation

$$2\mathrm{i}\varepsilon u_1\phi_x\phi_t + 0 \cdot v_1 = v_{0x}\phi_t + v_{0t}\phi_x - v_0\phi_{xt}. \tag{11}$$

From (11), we find that the function v_1 is arbitrary, whereas u_1 can be uniquely determined. Again by equating the coefficients of (ϕ^{-2}, ϕ^{-2}) in (4) we obtain a single equation

$$\begin{aligned}4\varepsilon v_0u_2\phi_x\phi_t + 2\mathrm{i}\varepsilon v_0v_2\phi_x\phi_t &= u_1(v_{0x}\phi_t + v_{0t}\phi_x + v_0\phi_{xt}) - \mathrm{i}\varepsilon v_0v_{0xt} - v_0u_{1t}\phi_x \\ &\quad - v_0u_{1x}\phi_t - \mathrm{i}\varepsilon v_0v_{1t}\phi_x - \mathrm{i}\varepsilon v_0v_{1x}\phi_t + 2\mathrm{i}\varepsilon v_{0x}v_{0t}\end{aligned} \tag{12}$$

so that the function u_2 (or v_2) is arbitrary. Thus the P property is satisfied and hence the system (4) is expected to be integrable. Also the complete integrability of (3) has already been pointed out (Chinea 1981) by means of a linear eigenvalue problem.

In order to find the Bäcklund transformation (BT) we truncate the series solution (8) up to a 'constant' level term, $u_j, v_j = 0, j \geq 2$. Thus we obtain the BT in the form

$$u = u_0\phi^{-1} + u_1, \qquad v = v_0\phi^{-1} + v_1, \tag{13}$$

where u_1 and v_1 satisfy (4) if u_0, v_0, u_1, v_1 and ϕ satisfy (11) and the following equations

$$\begin{aligned}u_1(v_{0x}\phi_t + v_{0t}\phi_x + v_0\phi_{xt}) - \mathrm{i}\varepsilon v_0v_{0xt} - v_0u_{1t}\phi_x - v_0u_{1x}\phi_t - \mathrm{i}\varepsilon v_0v_{1t}\phi_x \\ - \mathrm{i}\varepsilon v_0v_{1t}\phi_x - \mathrm{i}\varepsilon v_0v_{1x}\phi_t + 2\mathrm{i}\varepsilon v_{0x}v_{0t} = 0,\end{aligned} \tag{14a}$$

$$u_1u_{0xt} + u_0u_{1xt} - u_{0x}u_{1t} - u_{1x}u_{0t} + v_{0x}v_{1t} + v_{1x}v_{0t} = 0, \tag{14b}$$

$$u_1v_{0xt} + v_{1xt}u_0 - v_{1x}u_{0t} - u_{1t}v_{0x} - u_{0x}v_{1t} - u_{1x}v_{0t} = 0. \tag{14c}$$

Here we wish to point out that (14b) and (14c) are obtained by comparing the coefficient (ϕ^{-1}, ϕ^{-1}) in the series (8) in conjunction with (4) as in the previous cases and further we applied the restriction $u_2 = u_3 = v_2 = v_3 = 0$ in the resulting equation.

Finally let us consider the effect of the 'vacuum' solution $u_1, v_1 = 0$ in the BT (13). When we take into account of this constraint in (12) it reduces to

$$2v_{0x}v_{0t} = v_0v_{0xt}. \tag{15}$$

Integrating (15) and using (7) we find the particular solution

$$v_0 = -\left(\int a(t)\,\mathrm{d}t + b(x)\right)^{-1} \tag{16a}$$

and so

$$u_0 = -\mathrm{i}\varepsilon\left(\int a(t)\,\mathrm{d}t + b(x)\right)^{-1}, \tag{16b}$$

where $a(t)$ and $b(x)$ are arbitrary functions. By appropriately choosing the arbitrary functions $a(t)$ and $b(x)$ in (16) the 'algebraic' soliton solution of (4) can be obtained.

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References

Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)
Chinea F J 1981 *Lett. Math. Phys.* **5** 419
Chudnovsky D V, Chudnovsky G V and Tabor M 1983 *Phys. Lett.* **97A** 268
Lakshmanan M and Kaliappan P 1983 *J. Math. Phys.* **24** 795
Lakshmanan M and Sahadevan R 1985 *Phys. Rev.* A **31** 861
Lakshmanan M and Tamizhmani K M 1981 *Appl. Sci. Res.* **37** 127
Lund F 1976 *Phys. Rev.* D **14** 1524
Pohlmeyer K 1976 *Commun. Math. Phys.* **46** 207
Steeb W-H, Kloke M and Spieker B-M 1984 *J. Phys. A: Math. Gen.* **17** L825
Ward R S 1984 *Phys. Lett.* **102A** 279
Weiss J 1984 *J. Math. Phys.* **25** 13
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522